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Fractional supersymmetric Liouville theory and the multi-cut matrix models

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abstract

We point out that the non-critical version of the k-fractional superstring theory can be described by k-cut critical points of the matrix models. In particular, in comparison with the spectrum structure of fractional super-Liouville theory, we show that (p,q) minimal fractional superstring theories appear in the $\mathbb{Z}_k$-symmetry breaking critical points of the k-cut two-matrix models and the operator contents and string susceptibility coincide on both sides. By using this correspondence, we also propose a set of primary operators of the fractional superconformal ghost system which consistently produces the correct gravitational scaling critical exponents of the on-shell vertex operators.

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1 Introduction

String theory plays important roles not only as a candidate of the fundamental theory of our universe but also as an interesting tool to investigate other physics areas, including nuclear physics and condensed matter physics. We can also expect that there would be a possibility that other kinds of string theories have hidden some interesting connections to various regimes of physics.

The string theory considered in this paper is *fractional superstring theory* of order k [1], a different kind of string theory whose worldsheet gauge symmetry is so called *fractional supersymmetry* [2–8]. Each number k gives a different kind of theory (the case of $k = 1$ (2) is nothing but the usual bosonic (super) string theories). There are several reasons for studying these theories. One might be a phenomenological reason to have a model of lower (or reasonable) critical dimensions less than ten [9,10]. Another can be a possibility of extending the spacetime spectrum to include different types of statistics [11–13] (This feature might be good for applying to some systems with different kinds of statistics). In addition, it is also interesting to deepen our understanding of the RNS superstring formulation from the viewpoint of this generalization of worldsheet conformal field theory. For instance, this would be helpful to understand several structures among spacetime and worldsheet, like their symmetries and statistics.

Roughly speaking, fractional superstring theory is obtained by replacing the worldsheet fermions in the superstring theory with the Zamolodchikov-Fateev parafermions [14]. Even though there is much progress in studying this system [1, 9–13, 15–20], several difficulties have prevented us from revealing its whole body and structure. The main difficulty comes from the special feature of this fractional supersymmetry: spin of the current is equal to some fractional number, $(k + 4)/(k + 2)$ $(k = 2, 3, \cdots)$, and the algebra of this current is of the non-local non-Abelian braiding type [15]. Consequently, these facts cause the complexity of the system and also make it also difficult to identify the appropriate ghost system.

In this paper, we will shed new light on this string theory from another promising approach which is known as non-critical strings and matrix models [21–56]. We point out that the non-critical version of fractional superstring theory has the matrix-model dual description, known as multi-cut matrix models [43]. This should be an important clue to the investigation since it enables us to extract not only perturbative information but also non-perturbative one (for example, the D-branes in fractional superstring theory). The existence of a consistent fractional superstring theory is quite significant because we totally lost the

reason why we can ignore the possibility of this string theory. Since there are infinitely many kinds of the fractional superstrings (which are essentially related to the classification of Kac-Moody algebra) [58], this implication opens a broad possibility of string theory.

In the rest of this section, we explain the basic idea of the correspondence between fractional superstring theory and multi-cut matrix models [59], which can be seen from the minimal string field formulation for the multi-cut matrix models [56, 60]. Let us first recall the two-cut-matrix-model case, which corresponds to type 0 superstring theory [49–51].

The important point of the correspondence with superstrings ($k = 2$) is the fact that the $\mathbb{Z}_2$ reflection transformation of the two-cut-matrix-model eigenvalues, $\lambda \to -\lambda$, corresponds to the RR charge conjugation of the D-branes [49–51, 53]. Thereafter, this structure was shown to be clarified by the minimal string field formulation [46–48] of this two-cut case [56].

In the minimal superstring field formulation [56], the FZZT and its anti-FZZT brane in this theory were identified with the two-component fermions $c_0^{(1)}(\zeta)$ and $c_0^{(2)}(\zeta)$. The charge conjugation is, therefore, equivalent to exchanging these two fermions:

$$\Omega: \quad c_0^{(1)}(\zeta) \quad \leftrightarrow \quad c_0^{(2)}(\zeta). \tag{1.1}$$

One way to see the connection with the superstring theory is to consider the bosonization field $\varphi_0^{(i)}(\zeta)$ ($c_0^{(i)}(\zeta) \equiv : e^{\varphi_0^{(i)}(\zeta)} :$), which turns out to be a string field of the FZZT-brane boundary state [48]. That is, the following re-expression of the string fields,

$$\varphi_0^{(i)}(\zeta) \equiv \varphi_0^{[0]}(\zeta) + (-1)^{(i-1)} \varphi_0^{[1]}(\zeta), \tag{1.2}$$

with respect to the behavior of the charge conjugation ($\Omega: \varphi_0^{[\mu]}(\zeta) \to (-1)^{\mu} \varphi_0^{[\mu]}(\zeta)$) can be interpreted as the decomposition into the NSNS and RR contributions of the boundary states in the CFT language,

$$\big|B; (\zeta, i)\big\rangle = \big|B; \zeta\big\rangle_{\text{NSNS}} + (-1)^{i-1} \big|B; \zeta\big\rangle_{\text{RR}}. \tag{1.3}$$

Note that the conservation law of the RR charge is related to the $\mathbb{Z}_2$ spin structure of the worldsheet fermion. So the existence of the $\mathbb{Z}_2$ charged D-branes is one of the indications of superstring theory.

In the same way, the k-cut matrix model can be described with the k-component fermions $c_0^{(i)}(\zeta)$ ($i = 1, 2, \cdots, k$), and they have the following $\mathbb{Z}_k$-charge-conjugation property [60]:

$$\Omega^n: \quad c_0^{(i)}(\zeta) \quad \to \quad c_0^{(i+n)}(\zeta) \qquad (n \in \mathbb{Z}), \tag{1.4}$$

with $c_0^{(i+k)}(\zeta) = c_0^{(i)}(\zeta)$. This $\mathbb{Z}_k$ charge conjugation is also related to the $\mathbb{Z}_k$ rotation of the eigenvalue, $\lambda \to e^{\frac{2\pi i}{k}n}\lambda$, of the multi-cut matrix models (See [56] or section 2.2).

Therefore it is also natural to re-express the bosonization fields $\varphi_0^{(i)}(\zeta)$ $(i = 1, 2, \cdots, k)$ as

$$\varphi_0^{(i)}(\zeta) \equiv \varphi_0^{[0]}(\zeta) + \omega^{i-1}\varphi_0^{[1]}(\zeta) + \cdots + \omega^{(i-1)(k-1)}\varphi_0^{[k-1]}(\zeta) \qquad (\omega \equiv e^{2\pi i/k}), \tag{1.5}$$

with respect to the behavior of the $\mathbb{Z}_k$ charge conjugation $(\Omega : \varphi_0^{[\mu]}(\zeta) \to \omega^{\mu}\, \varphi_0^{[\mu]}(\zeta))$. Then the D-branes in the corresponding string theory should be expressed as the $\mathbb{Z}_k$ charged boundary states with *the $\mathbb{Z}_k$ generalized Ramond sector*:

$$\big|B; (\zeta, i)\big\rangle = \big|B; \zeta\big\rangle_{\mathrm{NSNS}} + \omega^{i-1}\big|B; \zeta\big\rangle_{\mathrm{RR}^{[1]}} + \cdots + \omega^{(i-1)(k-1)}\big|B; \zeta\big\rangle_{\mathrm{RR}^{[k-1]}}. \tag{1.6}$$

In this sense, it is natural to conjecture that the k-cut extension is realized by replacing the fermions in superstring theory with *parafermions* which realize the $\mathbb{Z}_k$ spin structure[1] on worldsheet [59]. That is, the multi-cut matrix models should correspond to the fractional superstring theory.[2] In particular, we will show that the simplest class of k-fractional superstring theory (i.e. (p, q) minimal k-fractional superstring theory) has the same spectrum structure and string susceptibility as those in the k-cut two-matrix model.

The organization of this paper is following: In section 2, we first give the definition and meaning of the multi-cut two-matrix models and then discuss the spectrum in the critical points. In section 3, we study fractional super-Liouville theory, especially the matching of the operator contents and string susceptibility in both sides. After that, we discuss the gravitational scaling exponents and the ghost primary fields in the fractional super-Liouville theory. Section 4 is devoted to conclusion and discussion. In the appendix A, we summarize the basics of parafermion and introduce some notations and terminology.

2 The multi-cut matrix models

2.1 The meaning of the multi-cut matrix integrals

Here we first note the meaning and definition of the multi-cut matrix model and then show its operator contents in next subsection.

[1] The basics of Zamolodchikov-Fateev parafermion and its $\mathbb{Z}_k$ spin-structure are in Appendix A.

[2] It was also pointed out in [59] that at least some special (but infinitely many) Kac tables of the unitary $(p, q) = (p, p + k)$ minimal fractional SCFT of the GKO coset construction [2] coincides with that of the differential operator system of the k-component KP hierarchy of k-cut matrix models.

The meaning of "multi-cut" in the multi-cut matrix models was first proposed by C. Crnkovid and G. Moore [43]. Since the double scaling limit means that we only focus on the vicinity of the critical points, the general multi-cut configuration should have the cuts which run in a radial pattern (See Figure. 1 (a)). This can be understood as "orbifold matrix models," which mean that the hermit matrix H is replaced with some matrix Φ such that $\Phi^k = H$ is hermit:[3]

$$\int DH\, e^{-N\,\mathrm{tr}\, V(H)} \quad \to \quad \int D\Phi\, e^{-N\,\mathrm{tr}\,\tilde{V}(\Phi)}. \tag{2.1}$$

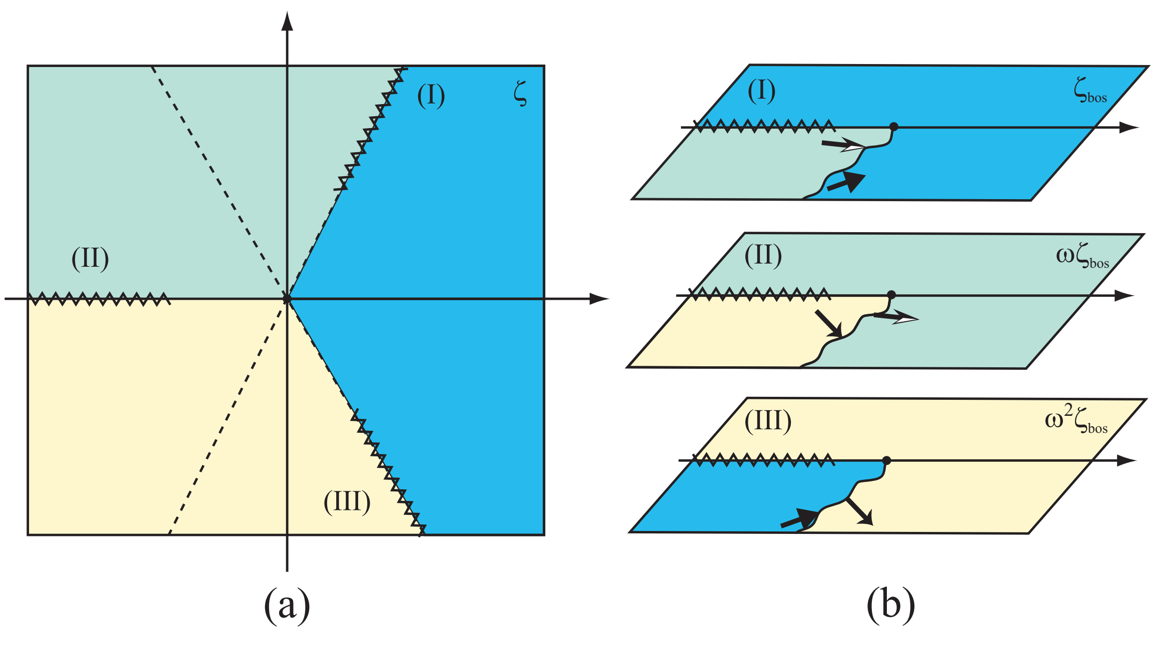

Figure 1: The case of $k = 3$: (a) The multi-cut geometry of the matrix models. The cuts form radial pattern. This means that there are $\mathbb{Z}_k$ mirror images of the spacetime. (b) The spacetime (Liouville-theory) geometry in terms of $\zeta_{bos} = \zeta^k$. There are k sheets (spacetime), and there is only one cut along the real axis on each sheet.

This orbifolding picture is also justified by the recent progress in the two-cut matrix models [51, 53, 56] which came after the original investigations [40–42, 44, 45]. In particular with the spacetime interpretation of the boundary cosmological constant $\zeta_{bos} = e^{-b(\phi+iX)}$ [48, 52, 53], it was argued by [56] that the multi-cut matrix models represent "orbifolding of

[3] Note that this configuration is different from the configuration where all the multi (more than two) cuts run only along the real axis in the planer limit of the matrix models, which is also called "multi-cut matrix models" in the literature.

spacetime" and that the multi-cut geometry of the multi-cut matrix models which is probed by cosmological constant ζ is the k-th root of the original spacetime, $\zeta_{bos}\,(=\zeta^k)$, on which there is only one cut along the real axis on each sheet (Figure 1 (b)). We can also consider some condensation of twisted modes in spacetime.[4] This is nothing but the consideration of the matrix-model potentials which breaks the $\mathbb{Z}_k$ symmetry [42].

The definition of multi-cut matrix models is usually given in the orthogonal polynomial method of the diagonalized matrix models since the essential information (the integrable hierarchy and string equations) comes from the recursive relations of the polynomials [43].

How to define multi-cut "multi-matrix" models was given in [56] as is shown below: The orthogonal polynomial system of the two-matrix models is as usual except for the contour $\mathcal{C}_k \times \mathcal{C}_k \subset \mathbb{C}^2$ of the integration of (x, y):

$$\langle \alpha_n | \, \beta_m \rangle \equiv \int_{\mathcal{C}_k \times \mathcal{C}_k} dxdy \, e^{-Nw(x,y)} \, \alpha_n(x) \, \beta_m(y) = \delta_{n,m}, \tag{2.2}$$

with the orthonormal polynomials,

$$\alpha_n(x) = \frac{1}{\sqrt{h_n}} x^n + \cdots , \quad \beta_n(y) = \frac{1}{\sqrt{h_n}} y^n + \cdots , \tag{2.3}$$

and the potential $w(x, y) = V_1(x) + V_2(y) - cxy$. The contour $\mathcal{C}_k \times \mathcal{C}_k \subset \mathbb{C}^2$ of the integration is

$$\mathcal{C}_k \equiv \big\{ \omega^n z \in \mathbb{C} \, ; z \in \mathbb{R}, \, n \in \mathbb{Z} \big\}, \tag{2.4}$$

with the parameter $\omega = e^{2\pi i/k}$. The proposal of how to construct the multi-cut matrix model [56] is: (i) to consider the following simultaneous Z_k transformation of eigenvalues (x, y):

$$\Omega^n : \, (x, y) \mapsto (\omega^n x, \omega^{-n} y) \qquad (n = 0, 1, 2, \cdots , k-1), \tag{2.5}$$

which should be respected in the k-cut models, and (ii) to focus on the critical behavior in the vicinity of the fixed point $(x, y) = (0, 0)$. In the special case of two-cut two-matrix models, this system correctly reproduces the results of super-Liouville theory, at least with some scaling ansatz [56].

One may wonder how we can realize this multi-cut geometry (especially $k \geq 3$) in terms of "matrix-model integral". That is, how to specify the matrix Φ and their measure $D\Phi$ in

[4] The R-R sector in the two-cut case.

its matrix model. It is easy to construct the matrix integral for multi-cut multi- (two- in practice) matrix models, and fortunately, it is the case where the matrix model should have the correspondence with minimal fractional superstrings. When we try to construct "multi-cut matrix quantum mechanics", we may find it is not so straightforward. This should be, however, just a technical problem because 2D fractional superstring theory would be consistently defined, at least from the the worldsheet point of views.[5]

The integral is given as follows: The important requirement here is that the measure of the inner product (2.2) should be invariant under this $\mathbb{Z}_k$ transformation, even though the matrix-model potentials $V_1(x)$ and $V_2(y)$ is not necessarily symmetric. Note that, with these requirements, the orthonormal conditions (2.2) are consistent under this $\mathbb{Z}_k$ transformation. From this orthogonal polynomial system, we can define the meaningful matrix integral,

$$\mathcal{Z}_{MM} = \int_{\mathcal{C}_{M_k}\times\mathcal{C}_{M_k}} dXdY\, e^{N\,\mathrm{tr}\, w(X,Y)}. \tag{2.6}$$

The measure is defined by the following metric:

$$ds_X^2 = \mathrm{tr}\big[(dX)^2\big], \qquad ds_Y^2 = \mathrm{tr}\big[(dY)^2\big], \tag{2.7}$$

and then the measure $dXdY$ is invariant under the $\mathbb{Z}_k$ transformation,[6]

$$(X,Y) \to (\omega^n X, \omega^{-n} Y). \tag{2.8}$$

The contour $\mathcal{C}_{M_k} \subset gl(N,\mathbb{C})$ of the matrix integral should be

$$\mathcal{C}_{M_k} \equiv \{U\,\mathrm{diag}(x_1,x_2,\cdots,x_N)\,U^\dagger\,;\, U\in U(N), x_i \in \mathcal{C}_k\} \subset gl(N,\mathbb{C}). \tag{2.9}$$

This means that the matrix X (and Y) of this model should be an $N\times N$ normal matrix X such that $X^{\hat{k}} = H$ is an hermit matrix.[7] Here we define,

$$\tilde{k} = \begin{cases} k & (k\in 2\mathbb{Z}+1) \\ k/2 & (k\in 2\mathbb{Z}). \end{cases} \tag{2.10}$$

Every odd k-cut model (i.e. k is odd) has the same contour integral as even $2k$-cut models have. They are, however, physically different systems because their Liouville directions $e^{-b\phi} = \mathrm{Re}\,(\zeta_{bos}) = \mathrm{Re}\,(\zeta^k)$ are different.

[5] Only when $k\le 2$, the two-matrix models can include the one-matrix models as the special case of the system because the Gaussian potential $V(y)=y^2$ which is necessary for the reduction into one-matrix model is forbidden in the higher $\mathbb{Z}_k$ symmetric case.

[6] We cannot use this measure when the number of the matrices in the model is odd, which includes one-matrix models and matrix quantum mechanics.

[7] This kind of matrix contour integral is also observed in the supermatrix description of the type 0 superstrings [57].

2.2 The spectrum of the multi-cut two-matrix models

In general, the matrix model possesses the integrable structure of KP hierarchy and the string field formulation [34, 35, 37–39, 46–48]. The two-cut matrix models also have the two-component extension of the KP hiererchy [56]. These structures are really useful to see the physical picture and dynamics of the string theory [46–48, 54, 55]. This formulation is also powerful even for the case of multi-cut matrix models and enable us to see various information of the matrix models.

From the case of two-cut matrix models [56], it is obvious that the integrable hierarchy of the k-cut matrix model is given by the k-component KP hierarchy, and the extension to the case of the k-component KP hierarchy is essentially the same as the extension to the two-component KP case [60].[8] So we just briefly mention how the k-component KP hierarchy appears in the k-cut two-matrix models to extract information of the spectrum. See [56] for more detail discussion, and [61] for the k-component KP hierarchy.

The integrable structure comes from the recursive relations of the orthonormal polynomials,

$$\begin{aligned} x\,\alpha_n(x) &= \hat{A}_x(n, e^{\partial_n}) \cdot \alpha_n(x), \qquad & N^{-1}\frac{\partial}{\partial x}\,\alpha_n(x) &= \hat{B}_x(n, e^{\partial_n}) \cdot \alpha_n(x), \\ y\,\beta_n(y) &= \hat{A}_y(n, e^{\partial_n}) \cdot \beta_n(y), \qquad & N^{-1}\frac{\partial}{\partial y}\,\beta_n(y) &= \hat{B}_y(n, e^{\partial_n}) \cdot \beta_n(y), \end{aligned} \tag{2.11}$$

with the canonical commutation relations, $[\hat{A}_x, \hat{B}_x] = [\hat{A}_y, \hat{B}_y] = N^{-1}$. For the simplicity, we first choose the $\mathbb{Z}_k$ symmetric potentials $V_1(\omega x) = V_1(x)$ and $V_2(\omega y) = V_2(y)$, then we can see the $\mathbb{Z}_k$ transformation property of the orthonormal polynomials:

$$\alpha_n(\omega x) = \omega^n \alpha_n(x), \qquad \beta_n(\omega y) = \omega^n \beta_n(y). \tag{2.12}$$

If we have critical points near the origin $(x, y) = (0, 0)$, then there should be the k kinds of scaling orthonormal functions $\{\hat{\Psi}_n(t;\zeta)\}_{n=1}^k$ which satisfy the same $\mathbb{Z}_k$ transformation property:

$$\hat{\Psi}_n(t;\omega\zeta) = \omega^n\,\hat{\Psi}_n(t;\zeta), \tag{2.13}$$

or equivalently, the scaling functions $\{\Psi_i(t;\zeta)\}_{i=1}^k$ which satisfy

$$\Psi_i(t;\omega\zeta) = \Psi_{i-1}(t;\zeta), \qquad \Psi_i(t;\zeta) = \sum_{n=0}^{k-1} \omega^{-ni}\,\hat{\Psi}_n(t;\zeta). \tag{2.14}$$

[8] The author would like to thank Prof. Masafumi Fukuma for the various useful discussion and sharing insights into this matter.

Furthermore, if we can find the special scaling limits,

$$x = \zeta\, a^{\frac{\hat{p}}{2}}, \qquad N^{-1} = g\, a^{\frac{\hat{p}+\hat{q}}{2}}, \qquad \frac{n}{N} = 1 + t\, a^{\frac{\hat{p}+\hat{q}-1}{2}}, \qquad \partial_n = (g\,\partial_t)\, a^{\frac{1}{2}}, \tag{2.15}$$

which make the operator $\hat{A}$ and $\hat{B}$ the differential operators $(\boldsymbol{P}, \boldsymbol{Q})$ in $\partial\,(\equiv \partial_t)$ of $(\hat{p}, \hat{q})$ order,[9]

$$\begin{aligned} a^{-\hat{p}/2}\hat{A}_x(n, e^{\partial_n}) \quad &\rightsquigarrow \quad \boldsymbol{P}(t,\partial) = U_0^{(P)}\partial^{\hat{p}} + U_1^{(P)}(t)\partial^{\hat{p}-1} + \cdots + U_{\hat{p}}^{(P)}(t), \\ a^{-\hat{q}/2}\hat{B}_x(n, e^{\partial_n}) \quad &\rightsquigarrow \quad \boldsymbol{Q}(t,\partial) = U_0^{(Q)}\partial^{\hat{q}} + U_1^{(Q)}(t)\partial^{\hat{q}-1} + \cdots + U_{\hat{q}}^{(Q)}(t), \end{aligned} \tag{2.16}$$

then the operator $(\boldsymbol{P}, \boldsymbol{Q})$ should be $k \times k$ matrix valued differential operators. At this level, we can consider $\mathbb{Z}_k$ breaking potential even such a critical points [42, 44, 45]. With these operators, we can see that the scaling recursive relations turn out to be the differential equations,

$$\zeta\, \Psi(t;\zeta) = \boldsymbol{P}(t,\partial)\, \Psi(t;\zeta), \qquad g\frac{\partial}{\partial\zeta}\Psi(t;\zeta) = \boldsymbol{Q}(t,\partial)\, \Psi(t;\zeta), \tag{2.17}$$

with $\Psi(t;\zeta) \equiv {}^t(\Psi_1, \Psi_2, \cdots, \Psi_k)$, which are related to the Baker-Akhiezer function with the Douglas equation $[\boldsymbol{P}, \boldsymbol{Q}] = g\boldsymbol{1}$. Since the operators are $k \times k$ matrix differential operators, we have k independent solutions $\{\Psi^{(i)}(t,\zeta)\}_{i=1}^k$,

$$\zeta_i\, \Psi^{(i)}(t;\zeta) = \boldsymbol{P}(t,\partial)\, \Psi^{(i)}(t;\zeta), \qquad g\frac{\partial}{\partial\zeta_i}\Psi^{(i)}(t;\zeta) = \boldsymbol{Q}(t,\partial)\, \Psi^{(i)}(t;\zeta). \tag{2.18}$$

Note that we take $\Psi^{(1)}(t,\zeta) = \Psi(t,\zeta)$ is a special solution $\zeta_1 = \zeta$ which satisfy (2.17). In the superstring case ($k = 2$), this another solution $\Psi^{(2)}$ is related to the anti-FZZT-branes which have different charges from the original $\Psi^{(1)}$. In particular, each solution $\Psi^{(i)}(t;\zeta)$ corresponds to each free fermion of the k-component KP hierarchy, $c_0^{(i)}(\zeta)$ (See [56] for how to relate). Thus it is natural to say that $\Psi^{(i)}$ (or $c_0^{(i)}(\zeta)$) corresponds to $\mathbb{Z}_k$ charged D-branes.

In general, the matrices $U_0^{(P)}$ and $U_0^{(Q)}$ commute with each other, thus they can be chosen to be diagonal matrices. Notice that the diagonal elements of $U_0^{(P)}$ are directly related to the eigenvalues ζ_i of each solution $\Psi^{(i)}(t,\zeta)$, and that we can freely change the relations among $\{\zeta_i\}_{i=1}^k$ and ζ.[10] That is, we can freely (without loss of generality) chose the matrix

[9] We should note that there should be several nontrivial changes of the basis of orthonormal polynomials which depend on the index n of $\alpha_n(x)$ to get the smooth scaling functions $\Psi_i(t;\zeta)$. For example, in the two-matrix model case, we need the change of the overall sign of the functions: $\alpha_n(x) \to (-1)^{[n/4]}\alpha_n(x)$ ($[*]$ is the Gauss symbol). Here we just assume the existence of scaling functions, which should be checked by some direct evaluation of critical points.

[10] It is because this is just a redefinition of the boundary cosmological constants $\{\zeta_i\}$ of each FZZT brane. See [56].

$U_0^{(P)}$ as[11]

$$U_0^{(P)} = \Omega \equiv \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{k-1} \end{pmatrix}, \tag{2.20}$$

which gives

$$\boldsymbol{P}(t,\partial)\,\boldsymbol{\Psi}(t,\zeta) = \boldsymbol{\Psi}(t,\zeta)\,Z, \qquad \boldsymbol{Q}(t,\partial)\,\boldsymbol{\Psi}(t,\zeta) = g\,\boldsymbol{\Psi}(t,\zeta)\,\overleftarrow{\frac{\partial}{\partial Z}}, \tag{2.21}$$

with

$$Z \equiv \begin{pmatrix} \zeta & & & \\ & \omega\,\zeta & & \\ & & \ddots & \\ & & & \omega^{k-1}\,\zeta \end{pmatrix}, \qquad \overleftarrow{\frac{\partial}{\partial Z}} \equiv \begin{pmatrix} \overleftarrow{\partial_\zeta} & & & \\ & \overleftarrow{\partial_\zeta}\,\omega^{-1} & & \\ & & \ddots & \\ & & & \overleftarrow{\partial_\zeta}\,\omega^{1-k} \end{pmatrix}, \tag{2.22}$$

and $\boldsymbol{\Psi}(t;\zeta) = (\Psi^{(1)}(t;\zeta), \Psi^{(2)}(t;\zeta), \cdots, \Psi^{(k)}(t;\zeta))$, an $N \times N$ matrix-valued function.[12]

With this differential operator $\boldsymbol{P}(x,\partial)$, we can construct the Lax operators of the k-component KP hierarchy, $\boldsymbol{L}$ and $\boldsymbol{\Omega}$,

$$\boldsymbol{L} = \partial + \sum_{n=0}^{\infty} U_n(t)\,\partial^{-n}, \qquad \boldsymbol{\Omega} = \Omega + \sum_{n=1}^{\infty} H_n(t)\,\partial^{-n}. \tag{2.23}$$

which satisfy $[\boldsymbol{L},\boldsymbol{P}] = [\boldsymbol{\Omega}^n,\boldsymbol{P}] = 0$ $(n = 1, 2, \cdots, k)$ and $\boldsymbol{P} = (\boldsymbol{\Omega}\boldsymbol{L}^{\hat{p}})$. In this sense, we can have the operators $\boldsymbol{Q}$ in terms of $\boldsymbol{L}$ and $\boldsymbol{\Omega}$ [39, 62]:

$$\boldsymbol{Q}(b;\partial) = \sum_{n=1}^{\hat{p}+\hat{q}} \sum_{\mu=0}^{k-1} \frac{n b_n^{[\mu]}}{\hat{p}} (\boldsymbol{\Omega}^{\mu-1}\boldsymbol{L}^{n-\hat{p}})_+ \tag{2.24}$$

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[11] The reason why this form is canonical can be easily seen as follows: For example, consider the $k = 3$ case. If we have the Z_k symmetric system, then the recursive relation of $\zeta\times$ (resp. $\partial_\zeta\times$) (2.17) should be a map from $\hat{\Psi}_n(t;\zeta)$ to $\hat{\Psi}_{n+1}(t;\zeta)$ (resp. from $\hat{\Psi}_n(t;\zeta)$ to $\hat{\Psi}_{n-1}(t;\zeta)$). Thus the differential operator $\boldsymbol{P}$ and $\boldsymbol{Q}$ should start from the shift matrices:

$$\boldsymbol{P} = \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix} \partial^{\hat{p}} + \cdots, \qquad \boldsymbol{Q} = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix} \partial^{\hat{q}} + \cdots, \tag{2.19}$$

the diagonalization of which gives (2.20).

[12] From these expression, we can easily see that, if we choose $\Psi^{(2)}$ as the starting point of the function (2.17), we need to change the boundary cosmological constant as $\zeta \to \omega^{-1}\zeta$. Then the role of each function $\Psi^{(i)}$ shifts as follows: $\Psi^{(i)} \to \Psi^{(i-1)}$. So there is no priory reason to choose $\Psi^{(1)}$ and this $\mathbb{Z}_k$ nature gives the "$\mathbb{Z}_k$ charge" of D-branes. (This is the same reason we can also assign a positive electric charge to electron.)

with the general background $b = \{b_n^{[\mu]}\}$. From these operators $(\boldsymbol{P}, \boldsymbol{Q})$, we can obtain the spectrum of the k-cut matrix models as the KP flow of this system [63]:

$$\boldsymbol{B}_n^{[\mu]} \equiv (\Omega^\mu \boldsymbol{L}^n)_+, \qquad g\frac{\partial}{\partial t_n^{[\mu]}}\boldsymbol{X}(t,\partial) = [\boldsymbol{B}_n^{[\mu]}, \boldsymbol{X}(t,\partial)], \tag{2.25}$$

where $\boldsymbol{X}(t,\partial) = \boldsymbol{P}(t,\partial), \boldsymbol{Q}(t,\partial), \boldsymbol{L}(t,\partial)$ and $\boldsymbol{\Omega}(t,\partial)$, and we abbreviate the KP times $\{t_n^{[\mu]}\}_{n,\mu}$ as $t = \{t_n^{[\mu]}\}_{n,\mu}$. The original t should be $t = t_1^{[0]}$. The special feature of the multi-component KP hierarchy is the $\mathbb{Z}_k$ indices μ of $t_n^{[\mu]}$ which indicate the behavior of the $\mathbb{Z}_k$ charge conjugation [56].

Therefore, the gravitational scaling exponents of this spectrum are

$$\boldsymbol{B}_n^{[\mu]} \sim a^{-n/2}, \tag{2.26}$$

with the lattice spacing a. The scaling of the most relevant operators $t_1^{[\mu]}$ and the "should-be" cosmological constant $\rho \equiv t_{\hat{q}-\hat{p}}^{[0]}$ are

$$t_1^{[\mu]} \sim a^{-\frac{\hat{p}+\hat{q}-1}{2}}, \qquad \rho \equiv t_{\hat{q}-\hat{p}}^{[0]} \sim a^{-\hat{p}}, \tag{2.27}$$

and the string susceptibility of the cosmological constant $\gamma_{str}^{(Mat)}$ is

$$\mathcal{F}_0(\rho) \sim \rho^{2-\gamma_{str}^{(Mat)}}, \qquad \gamma_{str}^{(Mat)} = 1 - \frac{\hat{q}}{\hat{p}}, \tag{2.28}$$

where $\mathcal{F}_0(\rho)$ is the scaling genus-zero free energy of the matrix model (or the partition function of the genus-zero worldsheet random surfaces), and the gravitational scaling dimension of each operator is given as

$$\left\langle \alpha_{n_1}^{[\mu_1]} \cdots \alpha_{n_l}^{[\mu_l]} \right\rangle_{\mathrm{c}} \equiv \frac{\partial^l \ln \mathcal{F}_0(t;\rho)}{\partial t_{n_1}^{[\mu_1]} \cdots \partial t_{n_l}^{[\mu_l]}}\Big|_{t=0,\rho\neq 0} \sim \rho^{\sum_{i=1}^l \frac{n_i - (\hat{p}+\hat{q})}{2\hat{p}}}. \tag{2.29}$$

Also as as important information of the differential operators $\boldsymbol{P}$ and $\boldsymbol{Q}$, the special KP flows of $\boldsymbol{P}^n$ and $\boldsymbol{Q}^n$ are trivial flows of the system, which should be unphysical-state propagations of the corresponding string theory [39]. In particular, since we have $\boldsymbol{P} = (\boldsymbol{\Omega}\boldsymbol{L}^{\hat{p}})$ as the canonical choice, the following operators

$$\boldsymbol{B}_{m\hat{p}}^{[m]} = \boldsymbol{P}^m = (\boldsymbol{\Omega}^m \boldsymbol{L}^{m\hat{p}}), \tag{2.30}$$

are trivial flows of the system, and especially if we consider $\mathbb{Z}_k$ symmetric critical points, we should have[13]

$$\boldsymbol{Q} = \sum_{n=1}^{\hat{q}+\hat{p}} \frac{n b_n^{[0]}}{\hat{p}} (\boldsymbol{\Omega}^{-1} \boldsymbol{L}^n) + \cdots, \tag{2.31}$$

[13]Diagonalization of of $\boldsymbol{Q}$ in (2.19) gives (2.31). See footnote 11.

and some flows of $\boldsymbol{B}^{[-n]}_{n\hat{q}} + \cdots$ should also be trivial.

Considering these information as inputs of the multi-cut matrix models, in next section, we will study the fractional super-Liouville theory and its comparison to the multi-cut matrix models.

3 Fractional super-Liouville theory

3.1 The Kac table of (p,q) minimal fractional SCFT

We first discuss comparison to the Kac table of the minimal fractional superconformal models, which can be derived from the generalized Feigin-Fuchs construction developed in [3,16]. The basic properties of parafermion and its terminology are summarized in Appendix A. The system is described with one free boson $X(z)$ with a background charge $\tilde{Q}$ and the Zamolodchikov-Fateev parafermion system Z_k. The action can be written as

$$S_{Matt} = \frac{1}{2\pi k}\int d^2z \Big(\partial X\bar{\partial}X + i\tilde{Q}\sqrt{g}\,RX\Big) + S_{Z_k}(\psi^M,\tilde{\psi}^M), \tag{3.1}$$

where $S_{Z_k}(\psi^M,\tilde{\psi}^M)$ is the action of the parafermion $\psi^M(z)$. Note that here we use the $\alpha' = k$ convention. The superscript M means that this is the FSUSY partner of matter bosonic field $X(z)$. The energy momentum tensor $T^M(z)$ and the fractional supercharge $G^M(z)$ is given as [1,8][14]

$$T^M(z) = -\frac{1}{k}\big(\partial X(z)\big)^2 + i\frac{\tilde{Q}}{k}\partial^2 X(z) + T^M_{Z_k}(z), \tag{3.2}$$

$$G^M(z) = \Big(\partial X(z) - i\frac{(k+2)\tilde{Q}}{4}\partial\Big)\epsilon^M(z) - i\frac{kQ}{k+4}\eta^M(z). \tag{3.3}$$

with the background charges $\tilde{Q}$ and Q, and the central charge c_M,

$$\tilde{Q} = b - \frac{1}{b}, \qquad Q = b + \frac{1}{b}, \qquad \hat{c}_M \equiv \frac{k+2}{3k}c_M = 1 - 2\frac{(k+2)}{k^2}\tilde{Q}^2. \tag{3.4}$$

The operator $\epsilon^M(z)$ is the first energy operator of matter parafermion and $\eta^M(z)$ is its first descendent in the sense of W_k-algebra [3]. The basic fractional super-primary operators[15] in this Feigin-Fuchs construction are given by

$$\mathcal{O}^{[\lambda]}_p(z) = \sigma_\lambda(z)\ :e^{i\frac{2}{k}pX(z)}:, \qquad \big(\mathcal{O}^{[\lambda]}_p\big)^\dagger(z) = \mathcal{O}^{[k-\lambda]}_{\tilde{Q}-p}(z), \tag{3.5}$$

[14] We used the same normalization for the primaries of the parafermion, ϵ and η, as in [1,14,15].

[15] This means that $G_r \cdot \mathcal{O}(z) = 0,\ (r>0)$, for the modes of the fractional supercurrent $G(z)$.

with the dimension

$$\Delta(\mathcal{O}_p^{[\lambda]}) = \Delta(\sigma_\lambda) - \frac{1}{k}p(\tilde{Q}-p), \qquad \Delta(\sigma_\lambda) = \frac{\lambda(k-\lambda)}{2k(k+2)}. \tag{3.6}$$

Here $\sigma_\lambda(z)$ is the spin field (See Appendix A). The Verma module of Feigin-Fuchs construction is generated by this primary:

$$\mathcal{V}_{p,\lambda} \equiv \left[\mathcal{O}_p^{[\lambda]}(z)\right]_{\epsilon^M, X} = \left[\mathcal{O}_p^{[\lambda]}(z)\right]_{G^M}. \tag{3.7}$$

The screening charge is defined with the basic parafermion $\psi(z)$ and $\psi^\dagger(z)$ as [3]

$$\mathcal{Q}_\pm \equiv \oint dz\, S_\pm(z), \qquad S_+ = \psi(z) :e^{2ibX(z)/k}:, \qquad S_- = \psi^\dagger(z) :e^{-2iX(z)/kb}:, \tag{3.8}$$

with $\Delta(S_\pm)=1$ and $\left[\mathcal{Q}_\pm, G^M(z)\right]=0$ [3]. The degenerate fields of the fractional superconformal field theory are given as [16]

$$\mathcal{O}_{r,s}(z) \equiv \mathcal{O}_{p_{r,s}}^{[r-s]}(z), \qquad p_{r,s} = -\frac{1}{2}\Big((1-r)\,b^{-1} - (1-s)\,b\Big). \tag{3.9}$$

Here we always write $[r-s]$ to indicate that $r-s$ should be understood as modulo k. The first null state in the module of $\mathcal{V}_{r,s}$ appears at the level of $N_{r,s}$ [16],

$$N_{r,s} = \frac{rs}{k} + \Delta(\sigma_{[r+s]}) - \Delta(\sigma_{[r-s]}), \tag{3.10}$$

as

$$\chi_{r,s}(z) = \Big(\mathcal{Q}_+^{(s)} \cdot \mathcal{O}_{-r,s}(z)\Big)^\dagger = \Big(\mathcal{Q}_-^{(r)} \cdot \mathcal{O}_{r,-s}(z)\Big)^\dagger \in \mathcal{V}_{r,s}. \tag{3.11}$$

The minimal model means that some finite set of degenerate primary operators are closed in the OPE algebra, and it is always realized if the parameter b is related to a rational number [64],

$$b = \sqrt{\frac{\hat{p}}{\hat{q}}}. \tag{3.12}$$

Here we assume that $(\hat{p},\hat{q})$ are coprime integers and $\hat{q} > \hat{p} > 0$.[16] Then the dimension of the primary operator is written as

$$\Delta(\mathcal{O}_{(r,s)}) = \frac{(\hat{q}r - \hat{p}s)^2 - (\hat{q}-\hat{p})^2}{4k\hat{p}\hat{q}} + \Delta(\sigma_{[s-r]}). \tag{3.13}$$

[16] Note that these indices are related to the exponent of differential operator in k-component KP hierarchy, so we used the same notation. One may want to write the labeling of minimal models by $(\hat{p},\hat{q};k)$ like in [43].

The usual conformal labeling of (p,q) is the range of the indices (r,s):

$$1 \le r \le p-1, \qquad 1 \le s \le q-1, \tag{3.14}$$

or we can also say that the minimal numbers (p,q) which satisfy

$$\mathcal{O}_{r+lp,s+lq}(z) = \mathcal{O}_{r,s}(z) \qquad (l \in \mathbb{Z}), \tag{3.15}$$

in the Feigin-Fuchs terminology. Because of this property, there appears additional basic null state:

$$\tilde{\chi}_{r,s}(z) = \mathcal{Q}_+^{(q-s)} \cdot \mathcal{O}_{-(p-r),(q-s)}(z) = \mathcal{Q}_-^{(p-r)} \cdot \mathcal{O}_{(p-r),-(q-s)}(z) \in \mathcal{V}_{r,s}, \tag{3.16}$$

at the level of

$$\tilde{N}_{r,s} = \frac{(p-r)(q-s)}{k} + \Delta(\sigma_{[q+p-r-s]}) - \Delta(\sigma_{[r-s]}), \tag{3.17}$$

and the labeling (p,q) appear in those formulas. One can easily see that this conformal labeling (p,q) is given as

$$(p,q) \equiv (\hat{k}\hat{p}, \hat{k}\hat{q}), \tag{3.18}$$

where $\hat{k}$ is defined by $k = \hat{k} \cdot d_{\hat{q}-\hat{p}}$ so that $d_{\hat{q}-\hat{p}}$ is the largest common divisor among the integers k and $\hat{q}-\hat{p}$.[17] Note that if we consider the special case where k is a prime number, then we have two kinds of models:

$$(p,q) = \begin{cases} (k\hat{p}, k\hat{q}) & : \hat{q}-\hat{p} \not\equiv 0 \pmod{k} \\ (\hat{p}, \hat{q}) & : \hat{q}-\hat{p} \equiv 0 \pmod{k}. \end{cases} \tag{3.19}$$

If we take $k=2$, for each case they are called even and odd models respectively. In this sense, this argument is consistent with the condition argued in [65]. Therefore the above (p,q) indices are the natural generalization of the constraints in minimal superconformal field theory ($k=2$), and there are several distinct kinds of minimal models in each k-th fractional superconformal field theory. The number of the kinds is give by the number of divisors of k plus one. Here we show some examples of the Kac table (Table 1 and Table 2 in the matrix-model language).

[17] This means that $q-p \equiv 0 \bmod. k$. We should note that another coprime labeling of the minimal model, (A,B), which was introduced in [16] to describe the fractional level coset model $SU(2)_k \otimes SU(2)_{A/B-2}/SU(2)_{k+A/B-2}$ of $(p,q) = (A, A+kB)$, is equivalent to our labeling (3.18).

					1
					ΩL^2
				ΩL	$\Omega^2 L^4$
			Ω	$\Omega^2 L^3$	L^6
			$\Omega^2 L^2$	L^5	ΩL^8
		$\Omega^2 L$	L^4	ΩL^7	$\Omega^2 L^{10}$
	Ω^2	L^3	ΩL^6	$\Omega^2 L^9$	L^{12}
	L^2	ΩL^5	$\Omega^2 L^8$	L^{11}	ΩL^{14}
L	ΩL^4	$\Omega^2 L^7$	L^{10}	ΩL^{13}	$\Omega^2 L^{16}$
ΩL^3	$\Omega^2 L^6$	L^9	ΩL^{12}	$\Omega^2 L^{15}$	L^{18}

Table 1: Kac table for $(\hat{p}, \hat{q}; k) = (2, 3; 3)$. Thus $(p, q) = (6, 9)$ and $d_{\hat{q}-\hat{p}} = 1$. The relation is $\mathcal{O}_{r,s} = (\boldsymbol{\Omega}^{r-s} \boldsymbol{L}^{\hat{q}r-\hat{p}s})_+$.

					1
					ΩL^3
				ΩL	$\Omega^2 L^6$
				$\Omega^2 L^4$	$\Omega^3 L^9$
			$\Omega^2 L^2$	$\Omega^3 L^7$	L^{12}
		Ω^2	$\Omega^3 L^5$	L^{10}	ΩL^{15}
		$\Omega^3 L^3$	L^8	ΩL^{13}	$\Omega^2 L^{18}$
	$\Omega^3 L$	L^6	ΩL^{11}	$\Omega^2 L^{16}$	$\Omega^3 L^{21}$
	L^4	ΩL^9	$\Omega^2 L^{14}$	$\Omega^3 L^{19}$	L^{24}
L^2	ΩL^7	$\Omega^2 L^{12}$	$\Omega^3 L^{17}$	L^{22}	ΩL^{27}
ΩL^5	$\Omega^2 L^{10}$	$\Omega^3 L^{15}$	L^{20}	ΩL^{25}	$\Omega^2 L^{30}$

Table 2: Kac table for $(\hat{p}, \hat{q}; k) = (3, 5; 4)$. Thus $(p, q) = (6, 10)$ and $d_{\hat{q}-\hat{p}} = 2$. There are two copies of Kac table. The relation is $\mathcal{O}_{r,s} = (\boldsymbol{\Omega}^{r-s} \boldsymbol{L}^{\hat{q}r-\hat{p}s})_+$, and the operators which do not correspond to the CFT one are obtained by just multiplying Ω: $\tilde{\mathcal{O}}_{r,s} = (\boldsymbol{\Omega}^{r-s+1} \boldsymbol{L}^{\hat{q}r-\hat{p}s})_+$.

From the Kac table, one can easily see that the labeling $(\hat{p},\hat{q})$ of the minimal models can be identified as the order of the differential operator $(\boldsymbol{P},\boldsymbol{Q})$ of the multi-cut matrix-model, and the relation with the KP flows is the following:

$$\mathcal{O}_{r,s} \quad \Leftrightarrow \quad \boldsymbol{B}^{[r-s]}_{n=\hat{q}r-\hat{p}s} \qquad (\hat{q}r-\hat{p}s\geq 0). \tag{3.20}$$

This correspondence indicates the matching of the unphysical spectrum,

$$\mathcal{O}_{p,q-n} \quad \Leftrightarrow \quad \boldsymbol{B}^{[n]}_{n\hat{p}}=(\boldsymbol{\Omega L}^{\hat{p}})^n, \qquad \mathcal{O}_{n,0} \quad \Leftrightarrow \quad \boldsymbol{B}^{[n]}_{n\hat{q}}=(\boldsymbol{\Omega L}^{\hat{q}})^n, \tag{3.21}$$

which means that the (p,q) minimal fractional superstring theory corresponds to the following operators:

$$\boldsymbol{P}=(\boldsymbol{\Omega L}^{\hat{p}}), \qquad \boldsymbol{Q}=\frac{(\hat{q}+\hat{p})b^{[2]}_{\hat{q}+\hat{p}}}{\hat{p}}\,(\boldsymbol{\Omega L}^{\hat{q}})_+ +\cdots\,, \tag{3.22}$$

in the multi-cut matrix models. Note that the operator $\boldsymbol{Q}$ does not start from $(\boldsymbol{\Omega}^{-1}\boldsymbol{L}^{\hat{q}})$ like (2.31) which was derived in the assumption of the $\mathbb{Z}_k$ symmetry of the matrix model. This means that the background corresponding to this minimal model basically *breaks* the original $\mathbb{Z}_k$ symmetry of the k-cut matrix model ($k\geq 3$), remaining at most $\mathbb{Z}_2$ symmetry of $(-1)^{r-s}$. This breaking symmetry property is actually an expected thing in the Liouville side, because the minimal-model correlators include the screening charges (3.8) which belong to $R^{[2]}$ sector (and $R^{[k-2]}$ sector as the dual) and also breaks the $\mathbb{Z}_k$ symmetry, remaining at most $\mathbb{Z}_2$ symmetry.[18] Note that the coincidence of this breaking nature is non-trivial since the origins of these phenomena in both sides are different.

Finally we should note that the matrix model includes some operators which do not correspond to the operators of the conformal field theory in general. This is also observed in the case of $k=2$ for the odd model and such operators could be dropped by gauging a $\mathbb{Z}_2$ symmetry [56]. Here we can drop these operators by gauging the following $\mathbb{Z}_k\times\mathbb{Z}_k$ symmetry,

$$(\boldsymbol{\Omega},\boldsymbol{L}) \quad \to \quad (\omega^{-(\hat{q}m-\hat{p}n)\hat{k}}\boldsymbol{\Omega},\omega^{(m-n)\hat{k}}\boldsymbol{L}) \qquad (m,n\in\mathbb{Z}), \tag{3.23}$$

and then the correspondence (3.20) become a one-to-one mapping. Interestingly, this $\mathbb{Z}_k\times\mathbb{Z}_k$ symmetry is essentially the symmetry which preserves the form of the differential operators $(\boldsymbol{P},\boldsymbol{Q})=(\boldsymbol{\Omega L}^{\hat{p}},(\boldsymbol{\Omega L}^{\hat{q}})_+)$. We also note that before gauging this symmetry there are $d_{\hat{q}-\hat{p}}$

[18]In this sense, the two-dimensional fractional superstring theory (which might correspond to multi-cut matrix quantum mechanics) should preserve the $\mathbb{Z}_k$ symmetry.

copies of the primary operators of $(p,q)=(\hat{k}\hat{p},\hat{k}\hat{q})$ minimal fractional SCFT and they are related by the action of Ω^l $(l=1,2,\cdots,d_{\hat{q}-\hat{p}}-1)$ on the gauge singlet operators.

As a summary, the correspondence with respect to the spectrum can be phrased as $(p,q)=(\hat{k}\hat{p},\hat{k}\hat{q})$ *minimal k-fractional superstring theory can be described with the* $\mathbb{Z}_k$ *breaking critical points of the k-cut two matrix model with* $\boldsymbol{P}=(\boldsymbol{\Omega L}^{\hat{p}})$ *and* $\boldsymbol{Q}=(\boldsymbol{\Omega L}^{\hat{q}})_+$. As we have seen, this correspondence requires several non-trivial matching of the spectrum structure. In the next subsection, we also consider coupling to the fractional super-Liouville system, and we will see that the string susceptibility of cosmological constant also coincides in both side.

3.2 Fractional super Liouville field theory

Here we discuss the critical exponents, so coupling to fractional super-Liouville field theory. The original construction of Liouville theory [21,23,24] always starts from the gauge fixing procedure of the (super-)diffeomorphism on worldsheet (super-)gravity. In the fractional super-Liouville case, we should also start from such a thing but we still do not know how to define "fractional supergravity" and even so-called "fractional superfield formalism" on "fractional superspace".

In the practical CFT calculation (for example [28]), we do not need the other terms which include contact terms [66].[19] So we assume that the action of fractional super Liouville field theory without contact terms is given as

$$S_{Liou}=\frac{1}{2\pi k}\int d^2z\Big(\partial\phi\bar{\partial}\phi+2\pi k\omega^{\frac{1}{2}}\,\rho\,\psi^L\tilde{\psi}^L\,e^{\frac{2}{k}b\phi}\Big)+S_{Z_k}(\psi^L,\tilde{\psi}^L), \tag{3.24}$$

in conformal gauge and we only consider this action. Here we also use the $\alpha'=k$ convention. This action is the Liouville counterpart of the fractional supersymmetric sine-Gordon theory [8]. Here $\phi(z,\bar{z})$ is the Liouville field, and $\psi^L(z)$ and $\tilde{\psi}^L(\bar{z})$ are the basic $\mathbb{Z}_k$ parafermion fields, with the statistics, $\tilde{\psi}^L(\bar{z})\psi^L(z)=\omega\,\psi^L(z)\tilde{\psi}^L(\bar{z})$. Let us call ρ the cosmological constant. At the first sight, this choice of the cosmological constant term is strange because this operator belongs to R^2 sector of the Liouville parafermion[20] and NS sector (i.e. identity operator) of the matter parafermion. However this choice is consistent with the string susceptability of the matrix model.

From this action, we can obtain the energy-momentum tensor $T^L(z)$ and the fractional

[19] For example, however, the minisuperspace formalism needs such terms [50].

[20] See Appendix A.

supercurrent $G^L(z)$ [1,8] as

$$T^L(z) = -\frac{1}{k}\big(\partial\phi(z)\big)^2 + \frac{Q_L}{k}\partial^2\phi(z) + T_{Z_k}(z),$$
$$G^L(z) = \Big(\partial\phi(z) - \frac{(k+2)Q_L}{4}\partial\Big)\epsilon(z) - \frac{k\tilde{Q}_L}{k+4}\eta(z), \tag{3.25}$$

with the background charges and the central charge,

$$Q_L = b_L + \frac{1}{b_L}, \qquad \tilde{Q}_L = b_L - \frac{1}{b_L}, \qquad \hat{c}_L \equiv \frac{k+2}{3k}c_L = 1 + 2\frac{(k+2)}{k^2}Q_L^2. \tag{3.26}$$

If this system couples to matter CFT, the background charges $(Q_L, \tilde{Q}_L)$ should be related to those of matter CFT, $(\tilde{Q}, Q)$ in (3.3). It is natural to consider the following ansatz:

$$Q_L \equiv Q = b + \frac{1}{b}, \qquad \tilde{Q}_L \equiv \tilde{Q} = b - \frac{1}{b}, \qquad b_L \equiv b. \tag{3.27}$$

This identification implies the critical central charge $\hat{c}_{crit}$ or the central charge of fractional superconformal ghosts, $\hat{c}_G$,

$$\hat{c}_{crit} \equiv \hat{c}_L + \hat{c}_M = 1 + \Big(\frac{k+4}{k}\Big)^2 = -\hat{c}_G. \tag{3.28}$$

This is actually nothing but the choice of the following critical central charge,

$$-c_G = -\frac{3k}{k+2}\hat{c}_G = \frac{6k}{k+2} + \frac{24}{k}, \tag{3.29}$$

which was found in [1]. So we just write $Q_L = Q$ and $\tilde{Q}_L = \tilde{Q}$ in the following discussion.

The basic primary operators are $V_\alpha^{[\mu]}(z) \equiv \sigma_\mu(z) : e^{\frac{2}{k}\alpha\phi(z)} :$ and their dual $(V_\alpha^{[\mu]}(z))^\dagger = V_{Q-\alpha}^{[k-\mu]}(z)$ with dimension:

$$\Delta(V_\alpha^{[\mu]}(z)) = \Delta(V_{Q-\alpha}^{[k-\mu]}(z)) = \Delta(\sigma_\mu) + \frac{1}{k}\alpha(Q-\alpha). \tag{3.30}$$

Since the Seiberg bound [25] in this case is also $\alpha \leq Q/2$, we choose the gravitational dressing [23] of each primary operators to satisfy this bound. The KPZ-DDK exponents [22–24] of the correlators can be calculated as in the usual way,

$$\left\langle \prod_{i=1}^{N} V_{\alpha_i}^{[\mu_i]}(z_i) \right\rangle_\rho = \rho^{\frac{Q}{2b}\chi - \sum_{i=1}^{N}\frac{\alpha_i}{b}} \left\langle \prod_{i=1}^{N} V_{\alpha_i}^{[\mu_i]}(z_i) \right\rangle_{\rho=1}, \tag{3.31}$$

where χ is the Euler number $\chi = 2 - 2h$ with the genus h.

From this relation, we can obtain the scaling relation of the genus-zero partition function with respect to cosmological constant ρ:

$$\mathcal{F}_0(\rho) \sim \rho^{2-(1-b^{-2})} \quad \Leftrightarrow \quad \gamma_{str}^{(Liou)} = 1 - b^{-2} = 1 - \frac{\hat{q}}{\hat{p}} = \gamma_{str}^{(Mat)}, \tag{3.32}$$

and this coincides with the matrix-model "should-be" cosmological-constant string susceptibility $\gamma_{str}^{(Mat)}$ of (2.28), with the identification of the pair of the coprime numbers $(\hat{p}, \hat{q})$ as the order of the differential operators $\boldsymbol{P}$ and $\boldsymbol{Q}$. This is also consistent with our identification of the operator contents given in section 3.1. Note that the consistent matching of string susceptibility depends on the choice of the cosmological constant term in the Liouville action (3.24) and the critical central charge of (3.27), (3.28) and (3.29).

3.3 The critical exponents and the on-shell vertex operators

For further evidence of the correspondence, we need to consider the gravitational exponents of all the on-shell vertex operators. Actually this is not so easy now because the corresponding ghost system has not been known until now, and other quantization procedures and GSO projection also include some mysterious things [9–13]. So we here give some proposal for the vertex operators and ghost primary fields which are consistent with the matrix models.[21] So this should be justified by some other procedures in future investigation.

From the critical central charge (3.29), the ghost system is expected to have the following central charge[22]

$$c_{Z_kG} \equiv c_G - c_{bc} = -2\,\frac{2(k-1)}{k+2} + 24\Big(1 - \frac{1}{k}\Big), \tag{3.33}$$

and to have a set of the "canonical" ghost primary fields $\Xi_\mu^{(\lambda)}(z)$ $(\lambda - \mu \in 2\mathbb{Z})$, with the identification,

$$\Xi_\mu^{(\lambda)}(z) = \Xi_{\mu+2nk}^{(\lambda)}(z) = \Xi_{\mu\pm nk}^{(\lambda+nk)}(z), \tag{3.34}$$

and the dimensions,

$$\Delta(\Xi_\mu^{(\lambda)}(z)) = 1 - \frac{1}{k} - \Delta(\sigma_\lambda) - \Delta(\sigma_\mu) \equiv 1 - a_\mu^{(\lambda)} \qquad \big(\lambda - \mu \in 2\mathbb{Z}\big). \tag{3.35}$$

[21] We should note that there is also another guess work for ghost system [11], which is different from ours.

[22] Here the system Z_kG indicates the "fractional super-partner" of the bc ghost system of $c_{bc} = -26$. Also note that this value has been also argued in [1, 10, 11].

Here we call them as "canonical" because we can expect that there should be some generalization of "picture" in the superstring $\beta\gamma$ ghost system [67] and that they belong to the canonical picture primary operators.[23] Also note that the value of $a_\mu^{(\lambda)}$ is the intercept $(L_0 - a_\mu^{(\lambda)})\big|phys\big\rangle = 0$ of the fractional super-Virasoro constraints for the sector which couples to the ghost primary field $\Xi_\mu^{(\lambda)}(z)$, especially $a_\lambda^{(\lambda+2)}$ is the value for the massless $\mathrm{R}^{[\lambda]}$ sector old covariant quantization discussed in [1, 20], which comes from the decoupling condition of the Lorentz-signature ghost contributions at the massless level.

With these ghost primary fields, we can write down the tachyon-level vertex operator as the following $(1,1)$-primary operators:

$$\mathcal{T}_{r,s}(z,\bar{z}) = \Xi_{r-s}^{(r+s)}(z,\bar{z})\, V_{r,-s}(z,\bar{z})\, \mathcal{O}_{r,s}(z,\bar{z}) \qquad (\hat{q}r - \hat{p}s \geq 0). \tag{3.36}$$

Here $V_{n,m}(z) \equiv V_{\alpha_{n,m}}^{[n-m]}(z)$ is the special primary field with

$$\alpha_{n,m} = \frac{1}{2}\Big[(1-n)b^{-1} + (1-m)b\Big]. \tag{3.37}$$

The Seiberg bound indicates the restriction $\hat{q}n + \hat{p}m \geq 0$.

If $mn > 0$, then $V_{n,m}(z)$ corresponds to degenerate fields in Liouville theory and the null state operator appears at the level $N_{n,m}$,

$$N_{n,m} = \frac{nm}{k} + \Delta(\sigma_{[n+m]}) - \Delta(\sigma_{[n-m]}), \tag{3.38}$$

and has the same dimension as $V_{-n,m}(z)$, that is, they are given as

$$\chi_{n,m}(z) = \Big(\mathcal{Q}_+^{(m)} \cdot V_{-n,m}(z)\Big)^\dagger = \Big(\mathcal{Q}_-^{(n)} \cdot V_{n,-m}(z)\Big)^\dagger \in \mathcal{V}_{n,m}. \tag{3.39}$$

It is worth to note the case of $b^2 = \hat{p}/\hat{q} \in \mathbb{Q}_+$. In this case, the range indices (p_L, q_L) of degenerate fields in Liouville theory appear, and they are the minimal integer pair of (p_L, q_L) satisfying

$$V_{n+lp_L, m-lq_L}(z) = V_{n,m}(z) \qquad (l \in \mathbb{Z}). \tag{3.40}$$

The distinct thing of this case is that they are *different* from the conformal labeling (p,q) of the matter CFT in general ($k \geq 3$), and should be chosen as

$$(p_L, q_L) = (\hat{k}_L \hat{p}, \hat{k}_L \hat{q}), \tag{3.41}$$

with $k = \hat{k}_L \cdot d_{\hat{q}+\hat{p}}$. Here $d_{\hat{q}+\hat{p}}$ is the maximal common divisor among k and $\hat{q}+\hat{p}$.

[23] For example, the cosmological constant term in the action (3.24) should belong to 0-picture.

There are several reason we choose the combination of the tachyon operators (3.36). The first reason is that these operators have the following gravitational scaling dimension:

$$\left\langle \prod_{i=1}^{l} \mathcal{T}_{r_i,s_i}(z_i) \right\rangle \sim \rho^{\sum_{i=1}^{l} \frac{n_i-(\hat{q}+\hat{p})}{2\hat{p}}} \qquad (n_i \equiv \hat{q}r_i - \hat{p}s_i \geq 0), \tag{3.42}$$

which is consistent with the scaling dimension in the matrix model (2.29), with the identification of (3.20): $\mathcal{O}_{r,s} \leftrightarrow \mathcal{T}_{r,s} \leftrightarrow \boldsymbol{B}^{[r-s]}_{\hat{q}r-\hat{p}s}$. Another important check is about the cosmological constant vertex operator,

$$\begin{aligned} \mathcal{T}_{1,1}(z,\bar{z}) &= \Xi_0^{(2)}(z,\bar{z})\, V_{1,-1}(z,\bar{z})\, \mathcal{O}_{1,1}(z,\bar{z}) \\ &= \Xi_0^{(2)}(z,\bar{z})\, \sigma_2^L(z,\bar{z})\, e^{\frac{2}{k}b\phi(z,\bar{z})}. \end{aligned} \tag{3.43}$$

This belongs to R^2 sector in the Liouville parafermion and has the same gravitational scaling dimension as that of the cosmological constant term in the action (3.24),

$$\psi^L(z)\tilde{\psi}^L(\bar{z})\, e^{\frac{2}{k}b\phi(z,\bar{z})}, \tag{3.44}$$

which also belongs to the same R^2 sector of the Liouville parafermion. From this we can argue the following thing:

Here we recall that the picture changing operator in the superstring case is given as the combination of the ghost field $\xi(z)$ and supercurrent (i.e. gauge current) [67]:

$$X \equiv \int dz\, \xi(z) G(z), \tag{3.45}$$

and that the fractional supercurrent $G(z)$ also belongs to NS sector whose action on other operators preserves the parafermion sector of the operators. This means that, even though the operators are expressed in the different pictures, the sector itself should *not* be changed. In this sense, our choice of the above cosmological constant operators in canonical picture ((3.43) and (3.36)) is consistent with the operator (3.44) in the action which can be related by some appropriate picture changing procedure.

The other reason is related to the decoupling conditions for the Lorentz-ghost contribution discussed in [1, 20]. The argument given in [1, 20] is essentially the consideration of $(p, q) = (2, k+2)$ pure fractional supergravity system in our terminology, and can be phrased as follows: Which dimension of the ghost primaries (i.e. the intercept a_λ) can enhance the null structure of the spectrum. The solution [1, 20] is

$$\begin{aligned} \chi_\lambda^{(a)}(z) &= \Xi_\lambda(z)\Big[\mathcal{Q}_+^{(\lambda+1+aq_L)} \cdot V_{-1-ap_L,\lambda+1+aq_L}(z)\Big]^\dagger \qquad (a = 0, 1, \cdots) \\ \tilde{\chi}_\lambda^{(a)}(z) &= \Xi_\lambda(z)\Big[\mathcal{Q}_-^{(aq_L-1)} \cdot V_{-1+ap_L,\lambda+1-aq_L}(z)\Big]^\dagger \qquad (a = 1, 2, \cdots), \end{aligned} \tag{3.46}$$

with $\Delta(\Xi_\lambda) = a_\lambda^{(\lambda+2)}$ of (3.35). That is, they are all $(1,1)$-primary operators and the Liouville primaries are null fields. Interestingly the matrix model implies that the set of ghost primary fields $\{\Xi_\lambda(z)\}_{\lambda=0}^k$ is not enough if we turn on the matter theory, and the generalization should be given as (3.36).

With the choice of the primary fields (3.36), there are the following enhancements of null structure:

$$\begin{aligned}
\chi_{r,s}^{(a)}(z) &= \Xi_{r-s}^{(r+s)}(z)\Big[\mathcal{Q}_-^{(r+ap_L)}\cdot V_{r+ap_L,-(s+aq_L)}(z)\Big]^\dagger \mathcal{O}_{r,s}(z),\\
\tilde{\chi}_{r,s}^{(\tilde{a})}(z) &= \Xi_{r-s}^{(r+s)}(z)\Big[\mathcal{Q}_+^{((\tilde{a}+1)q_L-s)}\cdot V_{-(p_L-r)-\tilde{a}p_L,(q_L-s)+\tilde{a}q_L}(z)\Big]^\dagger \mathcal{O}_{r,s}(z),
\end{aligned} \tag{3.47}$$

with $a,\tilde{a}=0,1,\cdots$. That is, they are all $(1,1)$-primary fields of the following level:

$$\begin{aligned}
N_{r,s;L}^{(a)} =& \frac{1}{k}(r+a\hat{k}_L\hat{p})(s+a\hat{k}_L\hat{q}) + \Delta(\sigma_{r+s}) - \Delta(\sigma_{r-s-a\hat{k}_L(\hat{q}-\hat{p})}),\\
\tilde{N}_{r,s;L}^{(\tilde{a}+1)} =& \frac{1}{k}(r-(\tilde{a}+1)\hat{k}_L\hat{p})(s-(\tilde{a}+1)\hat{k}_L\hat{q}) + \Delta(\sigma_{r+s}) - \Delta(\sigma_{r-s+(\tilde{a}+1)\hat{k}_L(\hat{q}-\hat{p})}).
\end{aligned} \tag{3.48}$$

We also note that the degenerate operator $\mathcal{O}_{r,s}$ has null states at the level

$$\begin{aligned}
N_{r,s;M}^{(a_M)} =& \frac{1}{k}(r+a_M\hat{k}\hat{p})(s+a_M\hat{k}\hat{q}) + \Delta(\sigma_{r+s+a_M\hat{k}(\hat{p}+\hat{q})}) - \Delta(\sigma_{r-s}),\\
\tilde{N}_{r,s;M}^{(\tilde{a}_M+1)} =& \frac{1}{k}(r-(\tilde{a}_M+1)\hat{k}\hat{p})(s-(\tilde{a}_M+1)\hat{k}\hat{q}) + \Delta(\sigma_{r+s-(\tilde{a}_M+1)\hat{k}(\hat{q}+\hat{p})}) - \Delta(\sigma_{r-s}),
\end{aligned} \tag{3.49}$$

and they turn out to be the same with the relations:

$$N_{r,s;L}^{(t\hat{k})} = N_{r,s;M}^{(t\hat{k}_L)}, \qquad \tilde{N}_{r,s;L}^{(t\hat{k})} = \tilde{N}_{r,s;M}^{(t\hat{k}_L)}, \qquad (t=1,2,\cdots). \tag{3.50}$$

Although this enhancement is not the same kind argued in [1,20], this should be related to the existence of the discrete states at the higher level [26].

4 Summary and discussion

In this paper, we have pointed out that the non-critical k-fractional superstring theory can be described by the k-cut matrix models. After we showed that multi-cut two-matrix model has the natural multi-cut matrix integral representation, we compared the operator contents of the matrix model and $(\hat{p},\hat{q})$ minimal k-fractional superconformal field theory. We then found that $(\hat{p},\hat{q})$ minimal k-fractional superstring theory corresponds to the critical point of k-cut matrix model with the differential operators $(\boldsymbol{P},\boldsymbol{Q}) = (\boldsymbol{\Omega}\partial^{\hat{p}}+\cdots,\boldsymbol{\Omega}\partial^{\hat{q}}+\cdots)$. Several consistency checks are in order:

- From the CFT point of view, the $\mathbb{Z}_k$ RR symmetry of the matter sector is broken by the screening charge and at most $\mathbb{Z}_2$ symmetry remains. We also observed this property in the matrix model side. That is, the critical point itself breaks the $\mathbb{Z}_k$ symmetry of the k-cut matrix models to at most $\mathbb{Z}_2$ symmetry.

- Although the matrix model includes much more operators than $(\hat{p}, \hat{q})$ minimal fractional superstrings, there is a $\mathbb{Z}_k \times \mathbb{Z}_k$ symmetry in the matrix model which can give the selection rule to correctly assign the operator contents of the CFT side.

- The definition of the cosmological constant operator is consistent with the matrix model, especially the string susceptibility with respect to the cosmological constant coincides on both sides.

From this coincidence, we can again make sure that the critical central charge (or ghost central charge) (3.29) argued in [1] should be the correct value. At least from these primary fields (3.35), the corresponding ghost system is like a "ghost parafermion system" which might be the wrong-statistics field theory of chiral (or "Weyl") parafermion.[24] It should be important to identify or concretely to construct the corresponding ghost CFT system.

Although there are several non-trivial checks in this paper, the coincidence of the gravitational exponents of "all" the vertex operators is not totally accomplished. In this sense, it should be fair to say that this correspondence is still at the level of conjecture. So other relevant checks (for example, the coincidence of the D-brane amplitudes in both sides) should be important. In particular, disk amplitudes [29] is interesting because this even does not require any knowledge of the ghost system. The interesting question is how the $\mathbb{Z}_k$ breaking nature does affect the $\mathbb{Z}_k$ charged FZZT brane and its algebraic curve. It is also interesting to compare the annulus amplitudes because annulus amplitudes are sensitive to the RR charges [30, 31].

Since the $\mathbb{Z}_k$-symmetry breaking critical points correspond to minimal k-fractional superstring theory, it is quite possible that there is some other $\mathbb{Z}_k$ symmetric minimal string theory which corresponds to the $\mathbb{Z}_k$ symmetric critical points of the k-cut two-matrix model. If there exists such a theory, the algebraic curve should show the k-cut geometry. Finding such a model should be also an interesting problem.

[24] The "wrong statistics" means that the statistics is different from the canonical one only by sign (-1). Also note that the usual parafermion system can be interpreted as "Majorana" parafermion because $k = 2$ gives Majorana fermion.

The important future work in the matrix-model direction is direct evaluation of the critical points, especially how we fix the "hermiticity" of the KP flow parameters $t_n^{[\mu]}$. In the two-cut case, this was identified in [45], that is, the odd potential should be pure imaginary. The multi-cut two-matrix model should also have such a selection of the potential and such a consideration can be only obtained from direct evaluation of the critical points.

In this paper, we only focus on minimal fractional superstrings which can be described by the two matrix model. An interesting direction is to give an answer to the question what should correspond to multi-cut one-matrix model, which was originally given by [43], and also how we can construct the multi-cut matrix quantum mechanics which should describe the two-dimensional fractional superstring theory.

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A The parafermion and the $\mathbb{Z}_k$ spin-structure

Here we summarize the basic ingredients and terminology for the parafermion CFT Z_k [14] which we use in the section 3, especially focusing on how they can be the extension of the usual fermion. This terminology should be useful for this kind of "parafermionic" string theory. The conformal field theory of parafermion is parametrized by integer $k = 1, 2, \cdots$ and has the central charge of $c_{Z_k} = 2(k-1)/(k+2)$. Basically this system is nothing but the coset CFT of $Z_k = SU(2)_k/U(1)_k$ [14]. Here we only focus on the left-hand-side (i.e.

chiral) part of the CFT, and detail discussions of the parafermion can be found in [14, 68].

A.1 Primary fields and the $\mathbb{Z}_k \times \tilde{\mathbb{Z}}_k$ parafermion charge

The basic dynamical degree of freedom is the parafermion field $\psi(z)$ with their descendants $\{\psi_l(z)\}_{l=0}^{k-1}$ of the dimensions,

$$\Delta_l \equiv \Delta(\psi_l) = \frac{l(k-l)}{k}, \tag{A.1}$$

with $\psi_0(z) = 1,\ \psi_1(z) = \psi(z),\ \psi_{k-l}(z) = \psi_l^\dagger(z)$ and $\psi_{l+nk}(z) = \psi_l(z)$ $(n \in \mathbb{Z})$. They form the following closed OPE algebra, the parafermion algebra:

$$\psi_{l_1}(z)\,\psi_{l_2}(0) \sim \frac{c_{l_1,l_2}}{z^{\Delta_{l_1}+\Delta_{l_2}-\Delta_{l_1+l_2}}}\big[\psi_{l_1+l_2}(0)+\cdots\big]. \tag{A.2}$$

The dots in the parenthesis indicate descendants fields in the sense of W_k-algebra [69]. So we often use the abbreviate notation:

$$\big[\psi_{l_1}(z)\big]_W \times \big[\psi_{l_2}(z)\big]_W = \big[\psi_{l_1+l_2}(z)\big]_W, \tag{A.3}$$

to clarify the algebraic structure. The important thing is that this algebra preserves the following $\mathbb{Z}_k$ symmetry:

$$\Gamma^n:\ \psi_l(z)\ \mapsto\ \omega^{nl}\psi_l(z), \qquad \Gamma \equiv \omega^{f_R}, \qquad \omega \equiv e^{\frac{2\pi i}{k}}. \tag{A.4}$$

Here we call f_R the worldsheet parafermion number (of left-hand-side). The charge Γ is the counterpart of the usual chirality operator and is called the (worldsheet) parafermion charge operator.

Other basic primary fields are spin fields $\{\sigma_\lambda\}_{\lambda=0}^{k}$ of the dimension

$$\Delta(\sigma_\lambda) = \frac{\lambda(k-\lambda)}{2k(k+2)}, \qquad \sigma_\lambda^\dagger(z) = \sigma_{k-\lambda}(z). \tag{A.5}$$

They are the vacuum operators of the parafermion module,

$$\big[\sigma_\lambda(z)\big]_\psi \equiv \bigoplus_{\mu\in\mathbb{Z}/\mathbb{Z}_k}\big[\sigma_\mu^{(\lambda)}(z)\big]_W, \qquad \big[\psi_l(z)\big]_W \times \big[\sigma_\mu^{(\lambda)}(z)\big]_W = \big[\sigma_{\mu-2l}^{(\lambda)}(z)\big]_W, \tag{A.6}$$

with

$$\sigma_\lambda(z) \equiv \sigma_\lambda^{(\lambda)}(z), \qquad \sigma_\mu^{(\lambda)}(z) = \sigma_{\mu+2nk}^{(\lambda)}(z) = \sigma_{\mu\pm nk}^{(\lambda+nk)}(z) \quad (n\in\mathbb{Z}), \tag{A.7}$$

especially $\psi_l(z) = \sigma_{2l}^{(0)}(z) = \sigma_{2l-k}^{(k)}(z)$. The dimension is

$$\Delta(\sigma_\mu^{(\lambda)}(z)) = \frac{\lambda(\lambda+2)}{4(k+2)} - \frac{\mu^2}{4k}. \tag{A.8}$$

Among them, there is the disorder (Kramers-Wannier) dual field $\mu_\lambda(z)$ for each $\sigma_\lambda(z)$:

$$\sigma_\lambda(z) \quad \leftrightarrow \quad \mu_\lambda(z) \equiv \sigma_{\lambda-k}^{(k-\lambda)}(z), \tag{A.9}$$

with $\mu_\lambda^\dagger(z) = \mu_{k-\lambda}(z)$. The assignment of the chirality Γ for each primary fields has two choices in general:

$$\Gamma(\sigma_\mu^{(\lambda)}(z)) = \omega^{\frac{\mu+\lambda}{2}}\sigma_\mu^{(\lambda)}(z), \qquad \tilde{\Gamma}(\sigma_\mu^{(\lambda)}(z)) = \omega^{\frac{\mu-\lambda}{2}}\sigma_\mu^{(\lambda)}(z), \tag{A.10}$$

especially $\Gamma(\sigma_\lambda) = \omega^\lambda = \tilde{\Gamma}(\mu_{k-\lambda})$ and $\tilde{\Gamma}(\sigma_\lambda) = 1 = \Gamma(\mu_{k-\lambda})$. In this sense, the parafermion basically has the $\mathbb{Z}_k \times \tilde{\mathbb{Z}}_k$ structure in its spectrum.

A.2 The $\mathbb{Z}_k$ spin-structure

Although the $\mathbb{Z}_k$ symmetry (A.4) is the symmetry of the algebra (A.3) and the module (A.6), this is not preserved in the general OPE algebra.[25] This $\mathbb{Z}_k$ degree of freedom however appears in the twisted boundary condition of parafermion fields:

$$\psi(e^{2\pi i}z)\,O_\mu(0) = \omega^\mu\,\psi(z)\,O_\mu(0), \qquad \Omega(O_\mu(z)) \equiv \omega^\mu O_\mu(z), \qquad \Omega \equiv \omega^F \tag{A.11}$$

and this $\mathbb{Z}_k$ charge Ω is preserved in the OPE algebra. This is the generalization of the spacetime fermion number or the R-R charge of the usual superstrings. We also call Ω the $\mathbb{Z}_k$ R-R charge or specetime "parafermion number", and we say that the operator O_μ belongs to R^μ sector, and especially R^0 sector is called NS sector. These sectors corresponds to the cuts of the parafermion fields in Riemann surfaces. We can easily see that the operator $\sigma_\mu^{(\lambda)}(z)$ belongs to R^μ sector. The special operators of NS sector, $\epsilon_i(z) \equiv \sigma_0^{(2i)}(z)$ ($\epsilon_i^\dagger(z) \equiv \sigma_k^{k-2i}(z)$), are called energy operators and have the dimension

$$\Delta(\epsilon_i(z)) = \frac{i(i+2)}{k+2}. \tag{A.12}$$

Note that this field has no $\mathbb{Z}_k$ R-R charge Ω but has the $\mathbb{Z}_k \times \tilde{\mathbb{Z}}_k$ parafermion charge $\Gamma \times \tilde{\Gamma}$.

We also note that it is convenient to use the Young diagram notation based on the $\mathbb{Z}_k \times \tilde{\mathbb{Z}}_k$ structure of (A.10):

$$\sigma_\mu^{(\lambda)}(z) \quad \Leftrightarrow \quad f_R\left\{ \ydiagram{2,2,1,1,1} \right. \left.\vphantom{\ydiagram{2,2}}\right\}\tilde{f}_R \quad \equiv {}^t(f_R, \tilde{f}_R), \tag{A.13}$$

[25]This charge is the coulomb charge of the Wakimoto construction and screened in general.

which is actually related to the W_k-labeling. In this notation, the conjugation of $\Gamma \leftrightarrow \tilde{\Gamma}$ expresses the Kramers-Wannier duality, and the other conjugation $(\Gamma, \tilde{\Gamma}) \to (\Gamma^\dagger, \tilde{\Gamma}^\dagger)$ means the dual field conjugation. The size of the diagram is nothing but the $\mathbb{Z}_k$ R-R charge: $\Omega = (\Gamma \times \tilde{\Gamma})_{diag}$.

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